TECHNICAL REPORT

HIGH-ORDER MIXTURE HOMOGENIZATION OF FIBER-REINFORCED COMPOSITES

by

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20. Abstract (continued)

while Reissner's mixed variational principle (1984, 1986), applied to the incremental boundary value problem yields consistent mixture constitutive relations. In order to assess the model accuracy, numerical experiments were conducted for static and dynamic loads. The prediction of the model in the time domain was obtained by an explicit finite element code. DYNA2D is used to furnish numerically exact data for the problems by discretizing the details of the microstructure. On the other hand, the model capability of predicting effective tangent moduli was tested by comparing results with NIKE2d. In all cases, good agreement was observed between the predicted and exact data for plastic, as well as, elastic responses.

Abstract

An asymptotic mixture theory of fiber-reinforced composites with periodic microstructure is presented for rate-independent inelastic responses, such as elastoplastic deformation. Key elements are the modeling capability of simulating critical interaction across material interfaces and the inclusion of the kinetic energy of microdisplacement. The construction of the proposed mixture model, which is deterministic, instead of phenomenological, is accomplished by resorting to a variational approach. The principle of virtual work is used for total quantities to derive mixture equations of motion and boundary conditions, while Reissner's mixed variational principle (1984, 1986), applied to the incremental boundary value problem yields consistent mixture constitutive relations. In order to assess the model accuracy, numerical experiments were conducted for static and dynamic loads. The prediction of the model in the time domain was obtained by an explicit finite element code. DYNA2D is used to furnish numerically exact data for the problems by discretizing the details of the microstructure. On the other hand, the model capability of predicting effective tangent moduli was tested by comparing results with NIKE2D. In all cases, good agreement was observed between the predicted and exact data for plastic, as well as, elastic responses.

Introduction

The increasing use of fiber-reinforced composites in the engineering technology requires the development of efficient models in order to properly simulate their static and dynamic behavior, such as effective moduli and wave dispersion effects. Furthermore, other applications may also involve load amplitudes that induce global material nonlinearities. Consequently, a nonlinear material model is needed.

The effective stiffness theory, which is a high-order continuum model, was developed by Achenbach and Herrmann (1968) to simulate elastic wave dispersion in fiber-reinforced composites. This theory has been later applied by Bartholomew and Torvick (1972), Hlavacek (1975), Achenbach (1976), and Aboudi (1981). By modifying the original methodology, Aboudi (1982, 1985) extended the linear model to account for visco-plastic material responses. In addition to the effective stiffness theory, mixture models have been developed by many investigators (Martin, Bedford and Stern, 1971; Choi and Bedford, 1973; Hegemier, Gurtman and Nayfeh, 1973; Hegemier and Gurtman, 1974; Nayfeh, 1977; Murakami, Maewal and Hegemier, 1979; Nayfeh, Crane and Hoppe, 1984; and Murakami and Hegemier, 1986). To date, high-order mixture models have not been extended to include nonlinear material responses for arbitrary wave motion.

In a highly heterogeneous medium, the large number of material interfaces renders the direct solution of the problem extremely complicated and time consuming. In order to alleviate such difficulties by deriving a set of partial differential equations with constant coefficients, two levels of homogenization methods have been introduced. They are: (1) O(1) homogenization methods (Babuska, 1976; Bensoussan, Lions, and Papanicolaou, 1978; Sanchez-Palencia, 1980; Tartar, 1977), and (2) an asymptotic mixture model based upon an O(ε) homogenization method (Murakami, Maewal, and Hegemier, 1981; Murakami, 1985; Murakami and Hegemier, 1986; Toledano and Murakami, 1987; Murakami and Toledano, 1988). O(1) methods, which yield

effective moduli, are non-dispersive. Therefore, simulation of wave dispersion requires an O(ε) (at least) homogenization model, where ε denotes a typical ratio of micro-to-macrodimension.

A high-order mixture homogenization model of fiber-reinforced composites with periodic microstructure is presented for rate-independent inelastic responses, such as elastoplastic deformation. The theory is based upon an asymptotic expansions with multiple scales. Key elements are the modeling capability of simulating critical material interaction and the inclusion of the kinetic energy of microdisplacements.

A variational approach is adopted in order to construct a mixture model, which is deterministic, instead of phenomenological. The principle of virtual work is used for total quantities to derive mixture equations of motion, while Reissner's mixed variational principles (1984, 1986) applied to rate boundary value problems are used to yield mixture constitutive relations. This is achieved by supplying the variational equations with appropriate trial displacement and transverse stress fields which must satisfy certain periodicity and normalization conditions. The accuracy of the resulting model clearly depends on the accuracy of the trial displacement and transverse stress fields, which may be found accurately as solutions of a system of rate-boundary value problems defined over a unit cell. This procedure has been successfully applied by Murakami and Toledano (1988) to predict transient responses of bi-laminated composites and is extended here to fiber-reinforced composites.

In order to assess the model accuracy, numerical experiments were conducted for static and dynamic loads. The prediction of the model in the time domain was obtained by an explicit finite element code. DYNA2D (Hallquist, 1982) is used to furnish numerically exact data for the problems by discretizing the details of the microstructure. On the other hand, the model capability of predicting effective tangent moduli was tested by comparing results with NIKE2D (Hallquist, 1986). In all cases, good agreement was observed between the predicted and exact data for plastic, as well as, elastic responses.

Formulation

Consider a body occupying a bounded domain Ω of R^3 with smooth boundary $\partial\Omega$, which contains fibers of circular cross section periodically distributed within the matrix, as shown in Fig. 1. The fibers occupy a volume $\Omega^{(1)}$ and the matrix a volume $\Omega^{(2)}$. A rectangular Cartesian coordinate system x_i is selected such that x_1 is parallel to the fibers. A periodic microstructure in the macrocoordinate plane (x_2, x_3) is defined by a cell, as shown in Fig. 2 for a hexagonal array.

The following notation: $(\ )^{(\alpha)}$, $\alpha=1,2$ will denote quantities associated with material α. Unless otherwise specified, the usual Cartesian indicial notation is employed where Latin indices range from 1 to 3 and Greek indices from 1 to 2. Repeated indices imply the summation convention. Also, $(\ )_{,i}$ and $(\ )_{,t}$ are used to denote partial differentiation with respect to x_i and time t, respectively.

With the help of the foregoing notation, the governing equations for the displacement vector $u^{(\alpha)}$ and stress tensor $\sigma^{(\alpha)}$ associated with the αth-constituent are:

(a) Equations of motion

$$\sigma^{(\alpha)}_{ji,j} + f^{(\alpha)}_i = \rho^{(\alpha)} u^{(\alpha)}_{i,tt} \quad , \quad \sigma^{(\alpha)}_{ji} = \sigma^{(\alpha)}_{ij} \qquad \text{in } \Omega^{(\alpha)} \tag{1}$$

where f_i is a constant body force and ρ denotes mass density;

(b) Rate constitutive relations

$$\dot{\sigma}^{(\alpha)}_{ij} = C^{(\alpha)(ep)}_{ijkl}\ \dot{e}^{(\alpha)}_{kl}(u^{(\alpha)}) \qquad \text{in } \Omega^{(\alpha)} \tag{2}$$

$$\dot{e}^{(\alpha)}_{kl}(u^{(\alpha)}) = \frac{1}{2}(\dot{u}^{(\alpha)}_{k,l} + \dot{u}^{(\alpha)}_{l,k}) \qquad \text{in } \Omega^{(\alpha)} \tag{3}$$

where $C^{(\alpha)(ep)}$ is the tangent modulus tensor;

(c) Interface continuity conditions

$$u_i^{(1)} = u_i^{(2)} \quad , \quad \sigma_{ji}^{(1)} v_j^{(1)} = \sigma_{ji}^{(2)} v_j^{(1)} \qquad \text{on } A_I \tag{4}$$

(d) Appropriate boundary conditions along $\partial\Omega$;

(e) Initial conditions on $\mathbf{u}^{(\alpha)}$ and $\mathbf{u}_{,t}^{(\alpha)}$ at t=0.

Conditions (a)-(e) define a well posed initial boundary value problem on Ω. However, due to the large number of heterogeneities in the medium, the direct solution of this problem represents a formidable task. To circumvent this difficulty, the method of homogenization which is based upon a multiscale representation and interior domain asymptotic expansions is adopted here. This procedure has been successfully applied by Murakami and Toledano (1989) for inelastic bi-laminated composites and is extended in the present work for fiber-reinforced composites.

Multivariable Field Representation

The material properties are periodic in the x-space, in which the periodicity is characterized by the cell. This representation implies that displacement and stress fields will vary according to two basic length scales: (i) a *macro* length characteristic of the body size, and (ii) a *micro* length characteristic of the cell spatial dimensions. Further, in most applications of interest, the microscale is much smaller than the macroscale. Therefore, it appears convenient to use an asymptotic technique based on the two scales involved in the problem. To this end new independent microcoordinates are introduced, such that

$$x_i^* = x_i / \varepsilon \quad , \qquad i=2,3 \tag{5}$$

where ε is a typical ratio of micro-to-macrodimension. As a consequence, all field variables now become functions of both the macrovariables $\mathbf{x}$ and microvariables $\mathbf{x}^*$

$$F(x_i, t) = F(x_i, x_j^*, t, \varepsilon) \tag{6}$$

Spatial derivatives of such a function F then take the form

$$F_{,i}(x_j, t) = F^*_{,i}\ (x_j, x_k^*, t, \varepsilon) + \frac{1}{\varepsilon} F^*_{,i^*}\ (x_j, x_k^*, t, \varepsilon) \tag{7}$$

where $(\ )_{,i^*}$ is used to denote partial differentiation with respect to x_i^*. For notational convenience, F^* will be written as F in what follows. It can be seen that macrovariables **x** account for slow variations of any material function, while microvariables **x*** account for rapid variations.

Applying (5) and (7) to eqns (1)-(4) yields a new set of "synthesized" governing equations, given by:

(a) Equations of motion

$$\sigma^{(\alpha)}_{ji,j} + \frac{1}{\varepsilon}\sigma^{(\alpha)}_{ji,j^*} + f^{(\alpha)}_i = \rho^{(\alpha)}\ u^{(\alpha)}_{i,tt} \qquad \text{in} \quad \Omega^{(\alpha)} \tag{8a}$$

$$\sigma^{(\alpha)}_{ji} = \sigma^{(\alpha)}_{ij} \qquad \text{in} \quad \Omega^{(\alpha)} \tag{8b}$$

(b) Constitutive relations

$$\dot{\sigma}^{(\alpha)}_{ij} = C^{(\alpha)\,(ep)}_{ijkl}\ \{\dot{e}_{kl}(u^{(\alpha)}) + \frac{1}{\varepsilon}\dot{e}^*_{kl}\,(u^{(\alpha)})\} \qquad \text{in} \quad \Omega^{(\alpha)} \tag{9}$$

$$\dot{e}^*_{kl}\,(u^{(\alpha)}) = \frac{1}{2}\,(\dot{u}^{(\alpha)}_{k,l^*} + \dot{u}^{(\alpha)}_{l,k^*}) \qquad \text{in} \quad \Omega^{(\alpha)} \tag{10}$$

(c) Interface continuity conditions

$$u^{(1)}_i = u^{(2)}_i \quad , \qquad \sigma^{(1)}_{ji}\,v^{(1)}_j = \sigma^{(2)}_{ji}\,v^{(1)}_j \qquad \text{on} \quad A_I \tag{11}$$

The hexagonal cell is modeled as two concentric cylinders, as shown in Fig. 2. Micropolar coordinates (r, θ) are introduced such that

$$r = \sqrt{{x_2^*}^2 + {x_3^*}^2} \quad , \qquad \tan\theta = x_3^* \ / x_2^* \tag{12}$$

With respect to these microcoordinates, eqn (12), a cell domain consists of subdomains $A^{(1)}$ and $A^{(2)}$ occupied by the fiber and matrix respectively, as shown in Fig. 2:

$$A^{(1)} = \{\ (r, \theta)\,|\,0 \le r \le \sqrt{n^{(1)}}\ ,\ 0 \le \theta \le 2\pi\ \}$$

$$A^{(2)} = \{\ (r, \theta)\,|\,\sqrt{n^{(1)}} \le r \le 1\ ,\ 0 \le \theta \le 2\pi\ \} \tag{13}$$

where $n^{(\alpha)}$ is the volume fraction and satisfies

$$n^{(1)} + n^{(2)} = 1 \tag{14}$$

Further, in the cell the interface is defined by $r=(n^{(1)})^{1/2}$ and the boundary by $r=1$. In terms of the polar coordinates (12), the **x***-periodicity condition for a hexagonal cell with the concentric cylinders approximation is expressed as

$$F(\mathbf{x}, r, \theta, t) = F(\mathbf{x}, r, \theta + \pi, t) \quad \text{at} \quad r = 1 \tag{15}$$

The **x***-periodicity condition implies that all field variables will assume equal values on opposite sides of the cell boundary. Therefore, it is only necessary to consider a single cell in order to determine the dependence of these field variables on the microcoordinates **x***. This assumption is adopted for elastic as well as inelastic deformations under the premise that a typical length of both elastic and inelastic regions is large compared to the cell dimension. This generalization is based upon the observation that, in an inelastic domain, tangent moduli of the two constituents may be well approximated by a locally periodic function of space variables. Since the tangent moduli depend also on the current stress state, exact periodicity conditions for tangent moduli might not hold, in general. However, under the premise that the variation of the tangent moduli with respect to the macrovariables is slow, inelastic tangent moduli may be approximated by a locally periodic function of space. Finally, it should be pointed out that synthesized displacement, stress and strain tensors are all continuous with respect to the macrovariables **x**, while retaining discrete heterogeneous structure with respect to the microvariables **x*** in each cell.

Variational Formulation

The construction of the present mixture model is accomplished by applying the principle of virtual work to the boundary value problem of total quantities, and Reissner's mixed variational principles (1984, 1986) to the incremental boundary value problem. This last operation allows one to derive consistent model constitutive equations.

The principle of virtual work applied to the fiber-reinforced composite yields

$$\int_{\Omega} \Big[\sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} \{ (\delta e^{(\alpha)T} + \frac{1}{\varepsilon} \delta e^{*(\alpha)T}) \sigma^{(\alpha)} - \delta u^{(\alpha)T} (f^{(\alpha)} - \rho^{(\alpha)} u^{(\alpha)}_{,tt}) \} \, dx^*$$

$$+ \frac{1}{\varepsilon} \int_{A_I} (\delta u^{(2)T} - \delta u^{(1)T}) T^* \, dA \Big] dx_1 dx_2 dx_3$$

$$= \int_{\partial\Omega_T} \Big[\sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} \delta u^{(\alpha)T} \, {}^{\nu}T^{(\alpha)} \, dx^* \Big] dA \qquad (16)$$

where $A^{(\alpha)}$ denotes the $\mathbf{x}^*$-domain occupied by constituent α, ${}^{\nu}T^{(\alpha)}$ represents the traction vector acting on the boundary ∂V_T with outward normal ν and $\mathbf{T}^*$ is the interface traction vector. For arbitrary variations of $\mathbf{u}^{(\alpha)}$, with (3) and (10) as definitions, the Euler-Lagrange equations of (16) include eqns (8) and appropriate boundary conditions.

The above variational equation (16) for total quantities yields any order of homogenized models with corresponding equations of motion by substituting therein appropriate trial displacements. However, for high-order homogenization models the associated rate-constitutive relations are obtained either by the application of Reissner's mixed variational principle (Reissner, 1984, 1986) or a set of experiments. In order to use Reissner's mixed variational principle, it is necessary to rewrite the incremental constitutive relations (9) in terms of in-plane strains and transverse stresses:

$$\dot{\sigma}_p = E_{11} \, \dot{e}_p + [E_{12}] \dot{\sigma}_t \qquad (17a)$$

$$\dot{e}_t = -[E_{12}]^T \dot{e}_p + [E_{22}] \dot{\sigma}_t \qquad (17b)$$

where

$$\sigma_t = [\sigma_{22} \; \sigma_{33} \; \sigma_{23} \; \sigma_{31} \; \sigma_{12}]^T, \quad \sigma_p = \sigma_{11}, \quad e_p = e_{11} \qquad (18a,b,c)$$

$$e_t = [\, e_{22} + e^*_{22} \quad e_{33} + e^*_{33} \quad 2e_{23} + 2e^*_{23} \quad 2e_{31} + 2e^*_{31} \quad 2e_{12} + 2e^*_{12} \,]^T \tag{18d}$$

Also $[\ ]^T$ is the transpose of $[\ ]$, subscripts p and t denote in-plane and transverse quantities respectively, and the matrices $[\, E_{\alpha\beta} \,]$ are functions of the elements of C_{ijkl}. Reissner's mixed variational principle applied to the incremental boundary value problem defined by (8)-(11) yields

$$\int_{\Omega} \Big[\sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} \{ \delta \dot{e}_p^{(\alpha)T} \dot{\sigma}_p^{(\alpha)} + \delta \dot{e}_t^{(\alpha)T} \dot{\sigma}_t^{(\alpha)} - \delta \dot{u}^{(\alpha)T} \rho^{(\alpha)} a^{(\alpha)}$$

$$+ \delta \dot{\sigma}_t^{(\alpha)T} (\dot{e}_t^{(\alpha)} + [\, E_{12}^{(\alpha)}]^T \dot{e}_p^{(\alpha)} - [\, E_{22}^{(\alpha)}] \dot{\sigma}_t^{(\alpha)}) \} \, d\, x^*$$

$$+ \frac{1}{\varepsilon} \int_{A_I} [(\delta \dot{u}^{(2)T} - \delta \dot{u}^{(1)T}) \dot{T}^* + \delta \dot{T}^{*T} (\dot{u}^{(2)} - \dot{u}^{(1)})] dA \Big] \, d x_1 \, d x_2 \, d x_3$$

$$= \int_{\partial\Omega_T} \{ \sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} \delta \dot{u}^{(\alpha)T} \, {}^{\nu}\dot{T}^{(\alpha)} \, d x^* \} \, dA \tag{19}$$

where $a^{(\alpha)}$ is the acceleration vector. The strains are all computed from the trial displacement field by (3) and (10). For arbitrary variations of $\dot{u}^{(\alpha)}$ and $\dot{\sigma}^{(\alpha)}$, the Euler-Lagrange equations of (19) yield the rate constitutive equations for $\dot{e}_t^{(\alpha)}$ in (17b), as well as the rate equations of motion, (8), the rate boundary conditions, (d), and the rate form of (11). Eqns (17a) are considered to be the definitions of $\dot{\sigma}_p^{(\alpha)}$.

The mixed variational equation (19) with appropriate trial functions $\dot{u}^{(\alpha)}$ and $\dot{\sigma}_t^{(\alpha)}$ yields the incremental constitutive relations for a given continuum model.

Asymptotic Analysis

The asymptotic technique starts by assuming the following expansions for the displacements, stresses and tangent moduli (Lene and Leguillon, 1982):

$$u^{(\alpha)}(x, x^*, t, \varepsilon) = u^{(\alpha)}_{(0)}(x, x^*, t) + \varepsilon\, u^{(\alpha)}_{(1)}(x, x^*, t) + \varepsilon^2 u^{(\alpha)}_{(2)}(x, x^*, t) + \ldots \quad (20a)$$

$$\sigma^{(\alpha)}(x, x^*, t, \varepsilon) = \frac{1}{\varepsilon}\sigma^{(\alpha)}_{(-1)}(x, x^*, t) + \sigma^{(\alpha)}_{(0)}(x, x^*, t) + \varepsilon\, \sigma^{(\alpha)}_{(1)}(x, x^*, t) + \ldots \quad (20b)$$

$$C^{(\alpha)\,(ep)} = C^{(\alpha)\,(ep)}_{(0)} + \varepsilon\, C^{(\alpha)\,(ep)}_{(1)} + \varepsilon^2 C^{(\alpha)\,(ep)}_{(2)} + \ldots \quad (20c)$$

where $u_{(n)}^{(2)}$, $\sigma_{(n)}^{(2)}$ and $C_{(n)}^{(2)}$ satisfy the x^*-periodicity condition. In the sequel, a class of inelastic materials, including hardening elastoplastic materials, which admit a rate potential and have positive definite tangent modulus tensor is considered. Therefore, in the expansion (20c), $C_{(0)}^{(\alpha)(ep)}$ is also assumed to be symmetric and positive definite, which obviously holds for elastic responses where only $C_{(0)}^{(\alpha)(ep)} = C^{(\alpha)}$ is required in the expansion.

Substituting (20a,b,c) into (8) - (11) and grouping terms in equal powers of ε, a sequence of microboundary value problems (MBVP's) defined over a unit cell is obtained.

MBVP for $O(\varepsilon^{-2})$:

$$\sigma^{(\alpha)}_{ji\,(-1)\,,\,j^*} = 0 \qquad \text{in} \quad A^{(\alpha)} \quad (21a)$$

$$\dot{\sigma}^{(\alpha)}_{ij\,(-1)} = C^{(\alpha)\,(ep)}_{ijkl\,(0)}\, \dot{e}^{*}_{kl}\,(u^{(\alpha)}_{(0)}) \qquad \text{in} \quad A^{(\alpha)} \quad (21b)$$

$$u^{(1)}_{i\,(0)} = u^{(2)}_{i\,(0)} \quad , \quad \sigma^{(1)}_{ji\,(-1)}\, v^{(1)}_{j} = \sigma^{(2)}_{ji\,(-1)}\, v^{(1)}_{j} \qquad \text{on} \quad A_I \quad (21c)$$

From (21a,b) the operator for $u_{(0)}^{(\alpha)}$ may be expressed as

$$\mathcal{L}(\dot{u}^{(\alpha)}_{(0)}) \equiv [\, C^{(\alpha)\,(ep)}_{ijkl\,(0)}\, \dot{e}^{*}_{kl}\,(u^{(\alpha)}_{(0)})\,]_{,j^*} = 0 \qquad \text{in} \quad A^{(\alpha)} \quad (22)$$

A solution of (22) is

$$u^{(\alpha)}_{(0)} = u_{(0)}(x, t) \quad , \quad e^*(u_{(0)}) = 0 \tag{23a}$$

$$\sigma^{(\alpha)}_{(-1)} = 0 \tag{23b}$$

Equation (23a) states that $u_{(0)}$ is independent of x^*.

MBVP for $O(\varepsilon^{-1})$:

$$\sigma^{(\alpha)}_{ji\,(0),\,j^*} = 0 \qquad \text{in} \quad A^{(\alpha)} \tag{24a}$$

$$\dot{\sigma}^{(\alpha)}_{ij\,(0)} = C^{(\alpha)\,(ep)}_{ijkl\,(0)}\,[\,\dot{e}_{kl}(u^{(\alpha)}_{(0)}) + \dot{e}^*_{kl}\,(u^{(\alpha)}_{(1)})\,] \qquad \text{in} \quad A^{(\alpha)} \tag{24b}$$

$$u^{(1)}_{i\,(1)} = u^{(2)}_{i\,(1)} \quad , \quad \sigma^{(1)}_{ji\,(0)}\,\nu^{(1)}_j = \sigma^{(2)}_{ji\,(0)}\,\nu^{(1)}_j \qquad \text{on} \quad A_I \tag{24c,d}$$

Eqns (24a,b) imply that

$$L(\dot{u}^{(\alpha)}_{(1)}) \equiv [\,C^{(\alpha)\,(ep)}_{ijkl\,(0)}\;\dot{e}^*_{kl}\;(u^{(\alpha)}_{(1)})\,]_{,\,j^*} = -\,C^{(\alpha)\,(ep)}_{ijkl\,(0),\,j^*}\;\dot{e}_{kl}(u^{(\alpha)}_{(0)}) \tag{25}$$

Eqn (25) shows that $u_{(1)}{}^{(\alpha)}$ is governed by the same operator as defined in (22), except for the RHS, which vanishes when integrated over the cell. Therefore the integrability condition for $u_{(1)}{}^{(\alpha)}$ is satisfied. The form of the forcing term in (25) suggests the following expression for $u_{(1)}{}^{(\alpha)}$

$$\dot{u}^{(\alpha)}_{i\,(1)}(x, x^*, t) = \dot{e}_{pq}(u^{(\alpha)}_{(0)})\,\chi^{pq}_i(x^*) \tag{26}$$

where $\chi_i{}^{pq}$ is x^*-periodic. The substitution of (26) into (25) yields a MBVP for each $\chi_i{}^{pq}$ which is continuous over the cell. These problems are defined up to a constant vector with respect to x^*.

Therefore, it is convenient to choose $\chi_i{}^{pq}$ such that its average over the cell vanishes

$$\chi^{pq(a)}_i \equiv \sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} \chi^{pq}_i(x^*)\,dx^* = 0 \tag{27}$$

MBVP for $O(1)$:

$$\sigma^{(\alpha)}_{ji(1),j^*} = \rho^{(\alpha)} u^{(\alpha)}_{i(0),tt} - f^{(\alpha)}_i - \sigma^{(\alpha)}_{ji(0),j} \qquad \text{in } A^{(\alpha)} \tag{28a}$$

$$\dot\sigma^{(\alpha)}_{ij(1)} = C^{(\alpha)(ep)}_{ijkl(0)} [\dot e_{kl} (u^{(\alpha)}_{(1)}) + \dot e^*_{kl} (u^{(\alpha)}_{(2)})]$$

$$+ C^{(\alpha)(ep)}_{ijkl(1)} [\dot e_{kl} (u^{(\alpha)}_{(0)}) + \dot e^*_{kl} (u^{(\alpha)}_{(1)})] \qquad \text{in } A^{(\alpha)} \tag{28b}$$

$$u^{(1)}_{i(2)} = u^{(2)}_{i(2)} \quad , \quad \sigma^{(1)}_{ji(1)} v^{(1)}_j = \sigma^{(2)}_{ji(1)} v^{(1)}_j \qquad \text{on } A_I \tag{28c.d}$$

The O(1) homogenization method consists in solving the MBVP's defined by eqns (24) and (26) for $\mathbf{u}_{(1)}^{(\alpha)}$. The O(1) equations of motion are obtained by posing the integrability condition on (28a). According to the Fredholm alternative theorem, the problem defined in (28) has a unique solution up to a constant vector with respect to $\mathbf{x}^*$, if the operator for $\mathbf{u}_{(2)}^{(\alpha)}$ in (28) satisfies the integrability condition, i.e the range of $\mathcal{L}$ ($\mathbf{u}_{(2)}^{(\alpha)}$) is orthogonal to its kernel $\mathbf{u}_{(0)}^{(\alpha)}$ $= \mathbf{u}_{(0)}$ ($\mathbf{x}$, t) (for example see Mardsen and Hughes, 1983; Toledano, 1989).

The O(1) equations of motion can also be obtained by substituting the trial displacement function

$$u^{(\alpha)}_i (x_j , x_k^*, t , \varepsilon) = u_{i(0)} (x_j , t) + \varepsilon S_{pq} (x_j , t) \chi^{pq}_i (x_k^*) \tag{29}$$

in eqn (16) and keeping only (1) terms, which implies that the kinetic energy due to the O(ε) displacement is not included. In order to improve the model capability in predicting dynamic responses of composites, it is natural to include the kinetic energy due to $\mathbf{u}_{(1)}^{(\alpha)}$. In addition, the O(ε) homogenization method requires the knowledge of the O(ε) transverse stresses by solving the MBVP defined in (28) for $\mathbf{u}_2^{(\alpha)}$, and then computing $\sigma_{(1)}^{(\alpha)}$ from (28b). However, due to the

complexity of the MBVP's an attempt is made to construct a reasonable approximation of $\sigma_{i(1)}{}^{(\alpha)}$ without solving for $u_2{}^{(\alpha)}$. The construction of the present O(ε) model is achieved by supplying the variational equations (16) and (19) with appropriate trial displacement and transverse stress fields which must satisfy the interface continuity condition (11) and the x*-periodicity condition on the cell boundary, eqn (15). As a result, only approximate solutions of the MBVP's are necessary. This procedure yields mixture equations of motion and consistent rate constitutive relations.

Trial Displacement and Transverse Stress Fields

One starts by defining average displacements for each constituent as:

$$U^{(\alpha)}(x,t) \equiv \frac{1}{V^{(\alpha)}} \int_{V^{(\alpha)}} \{ u_{(0)} + \varepsilon u^{(\alpha)}_{(1)} + \varepsilon^2 u^{(\alpha)}_{(2)} \} \, dx^* \tag{30}$$

Eqn (26) shows that $u_{(1)}{}^{(\alpha)}$ is excited by $u_{p(0),q}$. Therefore, the mixture formulation becomes more tractable by introducing microdisplacement variables which represent $u_{p(0),q} + u_{q(0),p}$, such that

$$S_{pq}(x,t) \equiv \frac{1}{\varepsilon V} \int_{A_I} u^{(\alpha)}_p v^{(1)}_q \, dA = \frac{1}{V} \int_{A_I} u^{(\alpha)}_{p(1)} v^{(1)}_q \, dA \tag{31a}$$

$$S_{pq} = S_{qp} \tag{31b}$$

The following trial displacement field may be used

$$u^{(\alpha)}_i(x_j, x_k^*, t, \varepsilon) = U^{(\alpha)}_i(x_j, t) + \varepsilon S_{pq}(x_j, t) \chi^{pq}_i(x_k^*) \tag{32}$$

In (32) $U^{(\alpha)}$ is the O(1) displacement associated with each constituent, while S_{pq} represents the amplitude of the O(ε) displacement microstructure. It is noted that, for general periodic composites, equations (32) imply that the resulting model includes *eleven* macrodisplacements

and corresponding equations of motion. Further, χ_i^{pq} depend, in general, on tangent moduli of the constituents and must therefore be evaluated at each load increment. However, good accuracy is achieved by assuming χ_i^{pq} to depend only on the microcoordinates $\mathbf{x}^*$. Such an approximation for the displacement microstructure, allows for the application of homogenization methods, even to nonlinear problems, without having to solve MBVP's for χ_i^{pq}.

For an elastic analysis, the exact solution of (24) for $\mathbf{u}_{(1)}^{(\alpha)}$ has been obtained by Murakami and Hegemier (1986). However, the present mixture analysis becomes more tractable by adopting an approximate form of the exact solution for $\mathbf{u}_{(1)}^{(\alpha)}$. The exact solution, together with (32), indicates that good accuracy of the O(ε) displacement microstructure is obtained by introducing five amplitude functions S_{ij} (i=1-3, j=2,3), and expressing the microstructure with two functions of r and θ. Eqn (32) then reduces to

$$u_i^{(\alpha)}(x, x^*, t, \varepsilon) = U_i^{(\alpha)}(x, t) + \varepsilon\,[\,S_{i2}(x, t)\cos\theta + S_{i3}(x, t)\sin\theta\,]\,g^{(\alpha)}(r) \quad (33a)$$

where

$$g^{(\alpha)}(r) = \frac{(-1)^{\alpha+1}}{n^{(\alpha)}}\left(r - \frac{\delta_{\alpha 2}}{r}\right) \quad (33b)$$

$$S_{23} = S_{32} \quad (33c)$$

In (33b) $\delta_{\alpha\beta}$ is the Kronecker delta. The functions $g^{(\alpha)}(r)\cos\theta$ and $g^{(\alpha)}(r)\sin\theta$ are continuous over the cell and satisfy the $\mathbf{x}^*$-periodicity condition (15) and normalization condition (27). The capability of the trial functions (33a) to simulate harmonic wave propagation in elastic fiber-reinforced composites has been demonstrated by Murakami and Hegemier (1986). Eqns (33) are used here to model nonlinear responses.

Substituting (33) into (24b) and considering the $O(\varepsilon^2)$ difference of the average transverse stresses, the following O(1) trial stress field is obtained

$$\dot{\sigma}^{(\alpha)}_{t\,(0)} = \dot{\tau}^{(\alpha)}(x,t) + \frac{\delta_{\alpha 2}}{r^2}\,[T(\theta)]\,\dot{t}\,(x,t) \tag{34a}$$

where

$$[T(\theta)] = \begin{bmatrix} \cos 2\theta & \cos 2\theta & \sin 2\theta & 0 & 0 \\ \cos 2\theta & -\cos 2\theta & \sin 2\theta & 0 & 0 \\ 0 & \sin 2\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 2\theta & \sin 2\theta \\ 0 & 0 & 0 & -\sin 2\theta & \cos 2\theta \end{bmatrix} \tag{34b}$$

In (34a), $\tau^{(\alpha)}$ and t are stress variables defined analogously to (18a). As an $O(\varepsilon)$ trial stress field one may use the following (Murakami and Hegemier, 1986):

$$\dot{\sigma}^{(\alpha)}_{t\,(1)} = \frac{\varepsilon}{4}\, g^{(\alpha)}(r)\,[Q(\theta)]\,\dot{P}(x,t) \tag{35a}$$

where

$$[Q(\theta)] = \begin{bmatrix} 0 & 3\cos\theta & \sin\theta \\ 0 & \cos\theta & 3\sin\theta \\ 0 & \sin\theta & \cos\theta \\ 2\sin\theta & 0 & 0 \\ 2\cos\theta & 0 & 0 \end{bmatrix} \tag{35b}$$

In (35a) P is an $O(\varepsilon)$ stress variable defined by

$$P_i(x,t) \equiv \frac{1}{\varepsilon V}\int_{A_I} \sigma^{(\alpha)}_{ji}\, v^{(1)}_j \, dA = \frac{1}{V}\int_{A_I} \sigma^{(\alpha)}_{ji\,(1)}\, v^{(1)}_j \, dA \tag{36}$$

Integrating (8a) over $A^{(\alpha)}$ and making use of the **x***-periodicity condition, the following mixture equation is obtained

$$n^{(\alpha)}\sigma^{(\alpha a)}_{ji,j} + (-1)^{\alpha+1} P_i + n^{(\alpha)} f^{(\alpha)}_i = n^{(\alpha)} \rho^{(\alpha)} U^{(\alpha)}_{i,tt} \tag{37}$$

where the average operation is defined by

$$F^{(\alpha a)}(\mathbf{x},t) \equiv \frac{1}{A^{(\alpha)}} \int_{A^{(\alpha)}} F^{(\alpha)}(\mathbf{x},\mathbf{x}^*,t)\, dx^* \tag{38}$$

As a result, the trial transverse stress field may now be written in the form

$$\dot{\sigma}^{(\alpha)}_t(\mathbf{x},\mathbf{x}^*,t,\varepsilon) = \dot{\tau}^{(\alpha)}(\mathbf{x},t) + \frac{\delta_{\alpha 2}}{r^2}[T(\theta)]\,\dot{t}(\mathbf{x},t) + \frac{\varepsilon}{4} g^{(\alpha)}(r)[Q(\theta)]\dot{P}(\mathbf{x},t) \tag{39}$$

Mixture Equations

Substituting (33) into (16) yields the mixture equations of motion, associated boundary and initial conditions, while the rate form of (33) and (39) into (19) yields the model rate constitutive relations. These equations are:

(a) Equations of motion

$$n^{(\alpha)}\sigma^{(\alpha a)}_{ji,j} + (-1)^{\alpha+1} P_i + n^{(\alpha)} f^{(\alpha)}_i = n^{(\alpha)} \rho^{(\alpha)} U^{(\alpha)}_{i,tt} \tag{40}$$

$$\overset{2}{M}_{ji,j} + \frac{1}{\varepsilon^2}(\sigma^{(2a)}_{2i} - \sigma^{(1a)}_{2i} + R_{2i}) = I S_{i2,tt} \quad , \quad i = 1,2 \tag{41a}$$

$$\overset{3}{M}_{ji,j} + \frac{1}{\varepsilon^2}(\sigma^{(2a)}_{3i} - \sigma^{(1a)}_{3i} + R_{3i}) = I S_{i3,tt} \quad , \quad i = 1,3 \tag{41b}$$

$$\frac{1}{2}(\overset{3}{M}_{j2,j} + \overset{2}{M}_{j3,j}) + \frac{1}{\varepsilon^2}(\sigma^{(2a)}_{23} - \sigma^{(1a)}_{23} + R_{23}) = I S_{23,tt} \tag{41c}$$

where

$$\varepsilon(\overset{2}{M}_{ij}, \overset{3}{M}_{ij}) \equiv \frac{1}{A} \sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} \sigma_{ij}^{(\alpha)} g^{(\alpha)} (\cos\theta, \sin\theta)\, d\, x^* \tag{42}$$

$$R_{2i} = \frac{1}{n^{(2)}A} \int_{A^{(2)}} \frac{1}{r^2} (\sigma_{2i}^{(2)} \cos 2\theta + \sigma_{3i}^{(2)} \sin 2\theta)\, d\, x^* \quad , \quad i = 1, 2 \tag{43a}$$

$$R_{3i} = \frac{1}{n^{(2)}A} \int_{A^{(2)}} \frac{1}{r^2} (-\sigma_{3i}^{(2)} \cos 2\theta + \sigma_{2i}^{(2)} \sin 2\theta)\, d\, x^* \quad , \quad i = 1, 3 \tag{43b}$$

$$R_{23} = \frac{1}{n^{(2)}A} \int_{A^{(2)}} \frac{1}{2r^2} (\sigma_{22}^{(2)} + \sigma_{33}^{(2)}) \sin 2\theta\, d\, x^* \tag{43c}$$

$$I \equiv \sum_{\alpha=1}^{2} h^{(\alpha)} \rho^{(\alpha)} \quad , \quad h^{(1)} = \frac{1}{4} \quad , \quad h^{(2)} = -\frac{1}{4n^{(2)}} (2 + n^{(2)} + \frac{2}{n^{(2)}} \ln n^{(1)}) \tag{44}$$

In (43) A (=π) denotes the area of the cell.

(b) Boundary conditions

$$\delta U_i^{(\alpha)} = 0 \qquad \text{OR} \qquad n^{(\alpha)} \sigma_{ji}^{(\alpha a)} v_j = T_i^{(\alpha p)} \tag{45}$$

$$\delta S_{i2} = 0 \qquad \text{OR} \qquad \overset{2}{M}_{ji} v_j = T_i^2 \quad , \quad i = 1,2 \tag{46a}$$

$$\delta S_{i3} = 0 \qquad \text{OR} \qquad \overset{3}{M}_{ji} v_j = T_i^3 \quad , \quad i = 1,3 \tag{46b}$$

$$\delta S_{23} = 0 \qquad \text{OR} \qquad (\overset{3}{M}_{j2} + \overset{2}{M}_{j3}) v_j = T_2^3 + T_3^2 \tag{46c}$$

where

$$T_i^{(\alpha p)} \equiv \frac{1}{A} \int_{A^{(\alpha)}} {}^{v}T_i^{(\alpha)}\, d\, x^* \tag{47a}$$

$$\varepsilon(T_i^2, T_i^3) \equiv \frac{1}{A} \sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} {}^{v}T_i^{(\alpha)} g^{(\alpha)} (\cos\theta, \sin\theta)\, d\, x^* \tag{47b}$$

(c) Initial conditions

$$U_i^{(\alpha)}\ ,\ U_{i,t}^{(\alpha)}\ ,\ S_{i2}\ ,\ S_{i3}\ ,\ S_{i2,t}\ ,\ S_{i3,t} \quad \text{at} \quad t=0 \tag{48}$$

(d) Constitutive relations

$$\int_{A^{(\alpha)}} [E_{22}^{(\alpha)}]\, dx^*\, \dot{\tau}^{(\alpha)} + \delta_{\alpha 2} \int_{A^{(\alpha)}} \frac{1}{r^2} [E_{22}^{(\alpha)}][T]\, dx^*\, \dot{t} + \frac{\varepsilon}{4} \int_{A^{(\alpha)}} g^{(\alpha)} [E_{22}^{(\alpha)}][Q]\, dx^*\, \dot{P}$$

$$= n^{(\alpha)} \pi \left(\dot{e}_{(U)t}^{(\alpha)} + \frac{(-1)^{\alpha+1}}{n^{(\alpha)}} \dot{S} \right) + \int_{A^{(\alpha)}} [E_{12}^{(\alpha)}]^T \dot{e}_{11}^{(\alpha)}\, dx^* \tag{49}$$

$$\int_{A^{(2)}} \frac{1}{r^2} [T]^T [E_{22}^{(2)}]\, dx^*\, \dot{\tau}^{(2)} + \int_{A^{(2)}} \frac{1}{r^4} [T]^T [E_{22}^{(2)}][T]\, dx^*\, \dot{t}$$

$$+ \frac{\varepsilon}{4} \int_{A^{(2)}} \frac{1}{r^2} g^{(2)} [T]^T [E_{22}^{(2)}][Q]\, dx^*\, \dot{P} = \pi [V] \dot{S}$$

$$+ \int_{A^{(2)}} \frac{1}{r^2} [T]^T [E_{12}^{(2)}]^T \dot{e}_{11}^{(2)}\, dx^* \tag{50}$$

$$\frac{\varepsilon}{4} \sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} g^{(\alpha)} [Q]^T [E_{22}^{(\alpha)}]\, dx^*\, \dot{\tau}^{(\alpha)} + \frac{\varepsilon}{4} \int_{A^{(2)}} \frac{1}{r^2} g^{(2)} [Q]^T [E_{22}^{(2)}][T]\, dx^*\, \dot{t}$$

$$+ \frac{\varepsilon^2}{16} \sum_{\alpha=1}^{2} \int_{A^{(\alpha)}} g^{(\alpha)2} [Q]^T [E_{22}^{(\alpha)}][Q]\, dx^*\, \dot{P} = \pi (\dot{U}^{(2)} - \dot{U}^{(1)})$$

$$+\pi\frac{\varepsilon^2 h}{4}\left([R_1]\dot{S}_{,1}+[R_2]\dot{S}_{,2}+[R_3]\dot{S}_{,3}\right)+\frac{\varepsilon}{4}\sum_{\alpha=1}^{2}\int_{A^{(\alpha)}} g^{(\alpha)}[Q]^T[E_{12}^{(\alpha)}]^T\dot{e}_{11}^{(\alpha)}\,dx^* \quad (51)$$

where

$$e_{(U)t}^{(\alpha)} = [\,U_{2,2}^{(\alpha)}\ \ U_{3,3}^{(\alpha)}\ \ U_{2,3}^{(\alpha)}+U_{3,2}^{(\alpha)}\ \ U_{3,1}^{(\alpha)}+U_{1,3}^{(\alpha)}\ \ U_{1,2}^{(\alpha)}+U_{2,1}^{(\alpha)}\,]^T$$

$$S = [\,S_{22}\ \ S_{33}\ \ 2S_{23}\ \ S_{13}\ \ S_{12}\,]^T \quad (52)$$

$$[V]=\frac{1}{n^{(1)}}\begin{bmatrix} -1/2 & 1/2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},\ [R_1]=\begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[R_2]=\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},\ [R_3]=\begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \end{bmatrix} \quad (53)$$

$\tau^{(\alpha)}$ and t are defined analogously to (18a). It is noted that for hardening materials, [$E_{22}^{(\alpha)}$] on the LHS of (49)-(51) is symmetric and positive definite. Specific forms of [$E_{ij}^{(\alpha)}$] for the von Mises yield criterion and associated flow rule with isotropic strain hardening are given in Appendix A. The solution of (49)-(51) yields $\dot{\tau}^{(\alpha)}$, $\dot{t}$, and $\dot{P}$ in terms of $\dot{e}_{(U)t}^{(\alpha)}$, $\dot{S}$, $\dot{e}_{11}^{(\alpha)}$, and $\dot{U}^{(2)}-\dot{U}^{(1)}$. For elastic constituents it is possible to evaluate the above relations explicitly (Murakami and Hegemier, 1986). However, for inelastic responses the integrals in (49)-(51) have to be evaluated numerically at each increment.

Substituting eqn (39) into (40) and (41) yields

$$\overset{2}{\dot{M}}_{22} = 3h\dot{P}_2/4 \;, \quad \overset{2}{\dot{M}}_{33} = \overset{3}{\dot{M}}_{23} = h\dot{P}_2/4 \;, \quad \overset{3}{\dot{M}}_{33} = 3h\dot{P}_3/4 \;,$$

$$\overset{2}{\dot{M}}_{12} = \overset{3}{\dot{M}}_{31} = h\dot{P}_1/2 \;, \quad \overset{3}{\dot{M}}_{22} = \overset{2}{\dot{M}}_{23} = h\dot{P}_3/4 \;,$$

$$\overset{2}{\dot{M}}_{31} = \overset{3}{\dot{M}}_{12} = 0 \;, \quad h = h^{(1)} + h^{(2)} \tag{54}$$

$$\dot{R}_{21} = \dot{t}_{12}/n^{(1)} \;, \quad \dot{R}_{22} = (\dot{t}_{22}/2 + \dot{t}_{33})/n^{(1)} \;, \quad \dot{R}_{31} = -\dot{t}_{31}/n^{(1)} \;,$$

$$\dot{R}_{23} = \dot{t}_{23}/2n^{(1)} \;, \quad \dot{R}_{33} = (-\dot{t}_{22}/2 + \dot{t}_{33})/n^{(1)} \tag{55}$$

The remaining constitutive equations for $\sigma_{11}^{(\alpha a)}$ and M_{11} are obtained from (17a), (38), (39) and (42). The results are

$$n^{(\alpha)}\pi\dot{\sigma}_{11}^{(\alpha a)} = \int_{A^{(\alpha)}} E_{11}^{(\alpha)}\,\dot{e}_{11}^{(\alpha)}\,dx^* + \int_{A^{(\alpha)}} [E_{12}^{(\alpha)}]\,dx^*\,\dot{\tau}^{(\alpha)}$$

$$+ \delta_{\alpha 2}\int_{A^{(\alpha)}} \frac{1}{r^2}[E_{12}^{(2)}][T]\,dx^*\,\dot{t} + \frac{\varepsilon}{4}\int_{A^{(\alpha)}} g^{(\alpha)}[E_{12}^{(\alpha)}][Q]\,dx^*\,\dot{P} \tag{56}$$

$$\varepsilon\pi\dot{M}_p = \sum_{\alpha=1}^{2}\int_{A^{(\alpha)}} g^{(\alpha)} n E_{11}^{(\alpha)}\,\dot{e}_{11}^{(\alpha)}\,dx^* + \sum_{\alpha=1}^{2}\int_{A^{(\alpha)}} g^{(\alpha)} n[E_{12}^{(\alpha)}]\,dx^*\,\dot{\tau}^{(\alpha)}$$

$$+ \delta_{\alpha 2}\int_{A^{(2)}} \frac{1}{r^2} g^{(2)} n[E_{12}^{(2)}][T]\,dx^*\,\dot{t} + \frac{\varepsilon}{4}\sum_{\alpha=1}^{2}\int_{A^{(\alpha)}} g^{(\alpha)2} n[E_{12}^{(\alpha)}][Q]\,dx^*\,\dot{P} \tag{57}$$

where

$$M_p = [\overset{2}{M}_{11} \;\; \overset{3}{M}_{11}]^T \;, \quad n = [\cos\theta \;\; \sin\theta]^T \tag{58}$$

The accuracy of the model in predicting elastic harmonic wave propagation has been amply demonstrated by Murakami and Hegemier (1986). Therefore, the accuracy of the model will be assessed by considering inelastic deformations under monotonic and step loading.

Reduction to an O(1) Model

The O(1) homogenized model can easily be obtained from the mixture model by introducing the kinematic constraints:

$$U_i^{(1)} = U_i^{(2)} = U_i \tag{59}$$

and taking the limit as $\varepsilon \to 0$. With (59), eqns (40) and (41) reduce to

$$\sigma_{ji,j}^{(m)} + f_i^{(m)} = \rho^{(m)} U_{i,tt} \tag{60}$$

$$\sigma_{2i}^{(2a)} - \sigma_{2i}^{(1a)} + R_{2i} = 0 \quad , \quad i=1\text{-}3$$

$$\sigma_{3i}^{(2a)} - \sigma_{3i}^{(1a)} + R_{3i} = 0 \quad , \quad i=1\text{-}3 \tag{61a,b}$$

where

$$\left\{ \sigma_{ij}^{(m)}, f_i^{(m)}, \rho^{(m)} \right\} = \sum_{\alpha=1}^{2} n^{(\alpha)} \left\{ \sigma_{ij}^{(\alpha a)}, f_i^{(\alpha)}, \rho^{(\alpha)} \right\} \tag{62}$$

The rate constitutive relations are obtained from (49)-(51) with (59) by setting ε=0. The results obtained from (49), (50) and (43) can be used to eliminate S_{ij} at each increment using (61). This procedure yields the incremental effective stress-strain relations given in Appendix B.

Numerical Results

The accuracy of the present model is tested by comparing results under monotonic and step loading with numerically exact data obtained from NIKE2D and DYNA2D. In what follows, material 1 is referred to the fiber and material 2 to the matrix.

For monotonic loading, the present O(1) model is tested and results compared with NIKE2D (Hallquist, 1986). A single transverse strain component is considered, say $U_{2,2}$. For the O(1) model, only one typical cell (Fig. 2) need be analyzed, while the details of the microstructure

are discretized in NIKE2D according to the mesh shown in Fig. 3. The material properties are given in Table 1. Fig. 4 shows the global response of the composite, where the effective transverse stress $\sigma_{22}^{(m)}$ is plotted as a function of the effective strain U_{22}. Excellent agreement is observed between the O(1) model and NIKE2D. Fig. 4 shows that the homogenized tangent moduli are $E_{22}^{(m)}$ = 12 GPa for the elastic range and $E_{22}^{(m)}$ = 8 GPa for the plastic range. Figs. 5 and 6 show the stress-strain curves at the center of the fiber and in the matrix at r=1, θ=0, respectively. Again, good agreement is obtained.

Next, the dynamic analysis consists of the wavereflect problem, illustrated in Fig. 7 wherein the same material properties are employed (see Table 1). The half space is subjected to a step pressure loading of intensity 1.4 GPa. The finite element mesh used in DYNA2D is identical to that shown in Fig. 3, except that now 20 cells are considered. The results of the present mixture model are obtained by subdividing each cell into 39 two-node, linear isoparametric elements for the elastic computation, and 11 elements for the elastoplastic case. The value of ε is $100\sqrt{2}$ microns. The time variations of the normal stress $\sigma_{22}^{(\alpha)}$ are computed at the center of the fiber, and in the matrix at r=1, θ=0 , of the eleventh cell from the impact surface. For an elastic analysis, results are shown in Fig. 8a for the fiber and Fig. 8b for the matrix, while for an elastoplastic analysis they are shown in Fig. 9a and Fig. 9b, respectively. It can be seen that a reasonable correlation is obtained between DYNA2D and the present mixture model.

In order to further assess the accuracy of the present mixture model, the waveguide problem illustrated in Fig. 10 is considered. Due the symmetry it is only necessary to analyze a single cell. The material properties are given in Table 2. The composite is subjected to a step pressure loading of 4.826×10^5 Pa (70 psi). The results of the mixture model are obtained by using 110 two-node, linear isoparametric elements. The cell dimension ε is 1.95 mm. The rear surface

velocities on a specimen 6.33 mm thick are compared with the experimental data recorded by Hegemier, Gurtman and Nayfeh (1973). Fig. 11 shows the time variation of the averaged velocity for an elastic analysis. In the case of an elastoplastic analysis the results of the present mixture model are compared with the numerical data presented by Hegemier and Gurtman (1974). The volume fractions are $n^{(1)}=0.16$ and $n^{(2)}=0.84$, while the cell dimension is $\varepsilon=4$mm. The average velocity histories at $x_1=5$mm in the fiber and matrix are shown in Fig. 12a and 12b, respectively. CRAM represents the results obtained by a two-dimensional finite difference code, while TINC represents the predicted values by Hegemier and Gurtman (1974). It can be seen that good agreement is obtained between the available data and the proposed mixture model.

Conclusion

A high-order mixture homogenization model of fiber-reinforced composites has been developed to simulate rate-independent inelastic responses, such as elastoplastic deformation. The construction of the mixture model was carried out by supplying the principle of virtual work for total quantities with an appropriate trial displacement field, and Reissner's mixed variational principle for the incremental boundary value problem with an appropriate trial transverse stress field. The former operation yielded equations of motion, boundary and initial conditions, while the latter yielded consistent model constitutive relations. It was found that eleven equations of motion were required for the mixture model. Good correlations with experimental and numerically exact data indicate that the proposed model may be used to simulate critical material interactions for static as well as dynamic responses.

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APPENDIX A - The Definitions of $[E_{ij}]$ in eqn (14)

It is computationally advantageous to rewrite eqn (9) in terms of the elastoplastic compliance matrix $D^{(\alpha)(ep)}$

$$\dot{e}^{(\alpha)} = D^{(\alpha)(ep)} \dot{\sigma}^{(\alpha)} \tag{A1}$$

For a von Mises yield criterion and associated flow rule with linear strain hardening, $D^{(\alpha)(ep)}$ may be expressed as

$$D^{(\alpha)(ep)} = D^{(\alpha)} + \frac{9}{4\bar{\sigma}^{(\alpha)2} H'^{(\alpha)}} s^{(\alpha)} s^{(\alpha)T} \tag{A2}$$

H' is the strain hardening parameter, μ is the shear modulus and the deviatoric stresses s and effective stress $\bar{\sigma}$ are respectively defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \tag{A3}$$

$$\bar{\sigma} = \sqrt{\frac{3}{2} s_{ij} s_{ij}} \tag{A4}$$

Using (A2), one finds

$$E_{11}^{(\alpha)} = \frac{1}{D_{11}^{(\alpha)(ep)}} \quad , \quad [E_{12}^{(\alpha)}] = -\frac{1}{D_{11}^{(\alpha)(ep)}} [\, D_{12} \;\; D_{13} \;\; D_{14} \;\; D_{15} \;\; D_{16} \,]^{(\alpha)(ep)}$$

$$[E_{22}^{(\alpha)}] = D_{ij}^{(\alpha)(ep)} - \left(\frac{D_{1i} D_{1j}}{D_{11}}\right)^{(\alpha)(ep)} , \quad i, j = 2\text{-}6 \tag{A5}$$

APPENDIX B - Effective stress-strain relations for fiber-reinforced composites.

The following stress-strain relations are obtained:

$$\begin{bmatrix} \dot{\sigma}_p^{(m)} \\ \dot{\sigma}_t^{(m)} \end{bmatrix} = \begin{bmatrix} D_{11}^{(m)} & [D_{12}^{(m)}] \\ [D_{12}^{(m)}]^T & [D_{22}^{(m)}] \end{bmatrix} \begin{bmatrix} \dot{U}_{1,1} \\ \dot{e}_{(U)t} \end{bmatrix} \tag{B1}$$

where

$$D_{11}^{(m)} = \sum_{\alpha=1}^{2} n^{(\alpha)} E_{11}^{(\alpha a)} + n^{(1)} [E_{12}^{(1a)}][E_{22}^{(1a)}]^{-1}[E_{12}^{(1a)}]^T + n^{(2)} \Big\{ [E_{12}^{(2a)}][M_1][E_{12}^{(2a)}]^T$$

$$+ [E_{12}^{(2a)}][M_2]\{a\} + \{a\}^T[M_2]^T[E_{12}^{(2a)}]^T + \{a\}^T[M_3]\{a\} \Big\} - \{b\}^T[D^{(n)}]\{b\} \qquad \text{(B2)}$$

$$[D_{12}^{(m)}] = n^{(1)}[E_{12}^{(1a)}][E_{22}^{(1a)}]^{-1} + n^{(2)} \Big\{ [E_{12}^{(2a)}][M_1] + \{a\}^T[M_2]^T \Big\}$$

$$- \{b\}^T[D^{(n)}][A] \qquad \text{(B3)}$$

$$[D_{22}^{(m)}] = n^{(1)}[E_{22}^{(1a)}]^{-1} + n^{(2)}[M_1] - [A]^T[D^{(n)}][A] \qquad \text{(B4)}$$

and

$$\{b\} = [M_1][E_{12}^{(2a)}]^T + [M_2]\{a\} - [E_{22}^{(1a)}]^{-1}[E_{12}^{(1a)}]^T$$

$$- [V]^T \Big\{ [M_2]^T[E_{12}^{(2a)}]^T + [M_3]\{a\} \Big\} \qquad \text{(B5)}$$

$$[A] = [M_1] - [E_{22}^{(1a)}]^{-1} - [V]^T[M_2]^T \qquad \text{(B6)}$$

$$[D^{(n)}] = \Big\{ \frac{1}{n^{(1)}}[E_{22}^{(1a)}]^{-1} + \frac{1}{n^{(2)}}\Big([M_1] + [V]^T[M_3][V]$$

$$- [M_2][V] - [V]^T[M_2]^T\Big) \Big\}^{-1} \qquad \text{(B7)}$$

$$[M_1] = [E_{22}^{(2a)}]^{-1}\Big([I] - [B_1][M_2]^T\Big), \quad [M_2] = -[E_{22}^{(2a)}]^{-1}[B_1][M_3],$$

$$[M_3] = \Big([B_2] - [B_1]^T[E_{22}^{(2a)}]^{-1}[B_1]\Big)^{-1} \qquad \text{(B8)}$$

$$[B_1] = \frac{1}{n^{(2)}\pi} \int_{A^{(2)}} \frac{1}{r^2} [E_{22}^{(2)}][T]\, dx^* \ , \ [B_2] = \frac{1}{n^{(2)}\pi} \int_{A^{(2)}} \frac{1}{r^4} [T]^T [E_{22}^{(2)}][T]\, dx^* \ ,$$

$$\{a\} = \frac{1}{n^{(2)}\pi} \int_{A^{(2)}} \frac{1}{r^2} [T]^T [E_{12}^{(2)}]^T dx^* \qquad \text{(B9)}$$

In eqn (B8a), [I] denotes the identity matrix. {a} and {b} are 5x1 column vectors, [E_{12}] and [$D_{12}^{(m)}$] are 1x5 row vectors and $D_{11}^{(m)}$ is a scalar. All other quantities are 5x5 matrices. In addition, matrices [$D_{22}^{(m)}$], [M_1] and [M_3] are symmetric.

Table 1 Material Properties for Wavereflect Problem.

Material α	Volume Fraction $n^{(\alpha)}$	Density $\rho^{(\alpha)}(kg/m^3)$	Young's Modulus $E^{(\alpha)}(GPa)$	Poisson's Ratio $\nu^{(\alpha)}$	Yield Stress $\sigma_Y(GPa)$	Hardening Parameter $H'(GPa)$
1	0.25	2200	69	0.17	—	—
2	0.75	1266	6.895	0.25	0.7	0.35

Table 2 Material Properties for Waveguide Problem.

Material α	Volume Fraction $n^{(\alpha)}$	Density $\rho^{(\alpha)}(kg/m^3)$	Young's Modulus $E^{(\alpha)}(GPa)$	Poisson's Ratio $\nu^{(\alpha)}$	Yield Stress $\sigma_Y(GPa)$
1	0.272	1850	29	0.38	—
2	0.728	1290	8.25	0.36	0.133

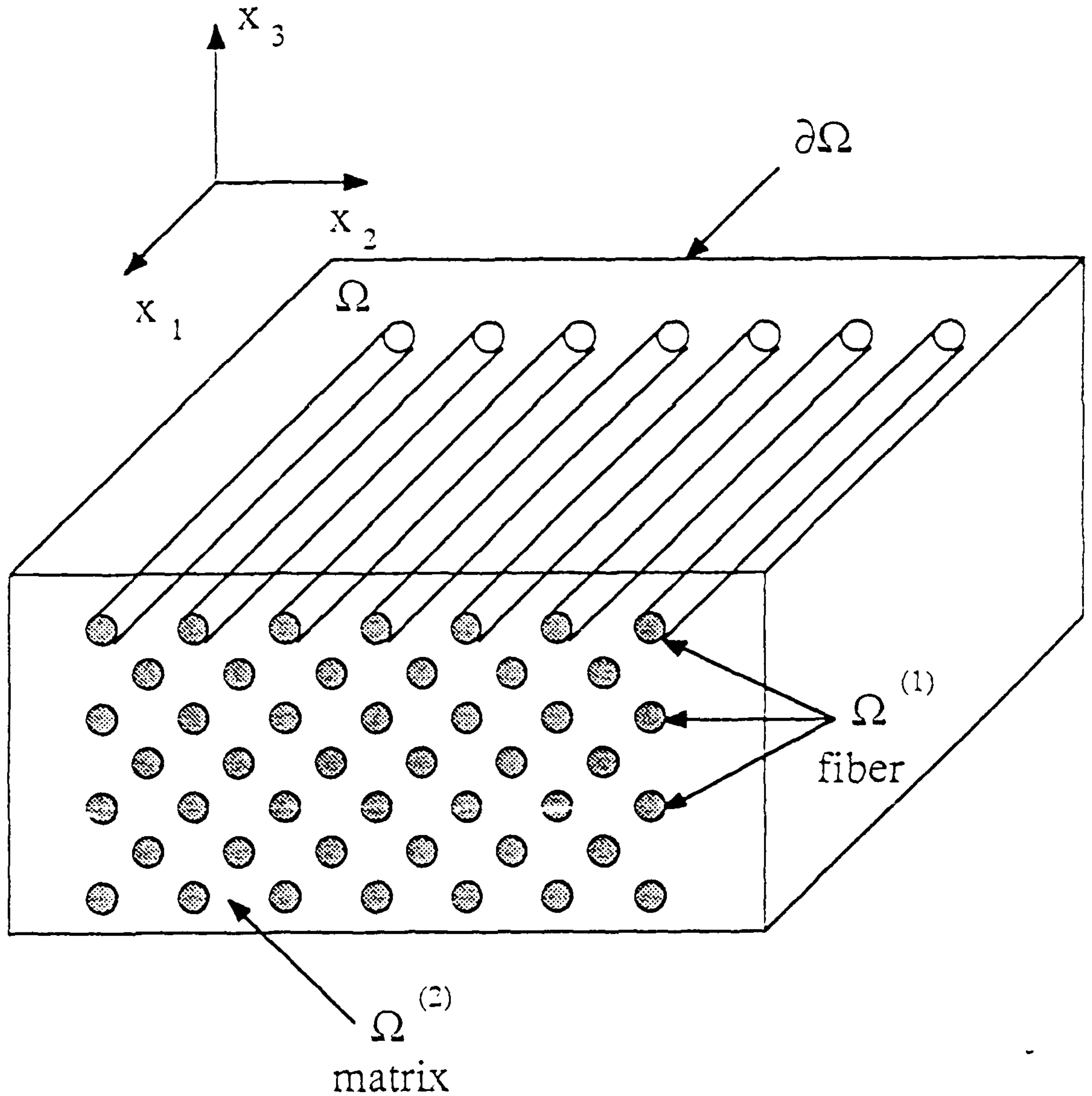

Fig. 1 A fiber-reinforced composite and the coordinate system.

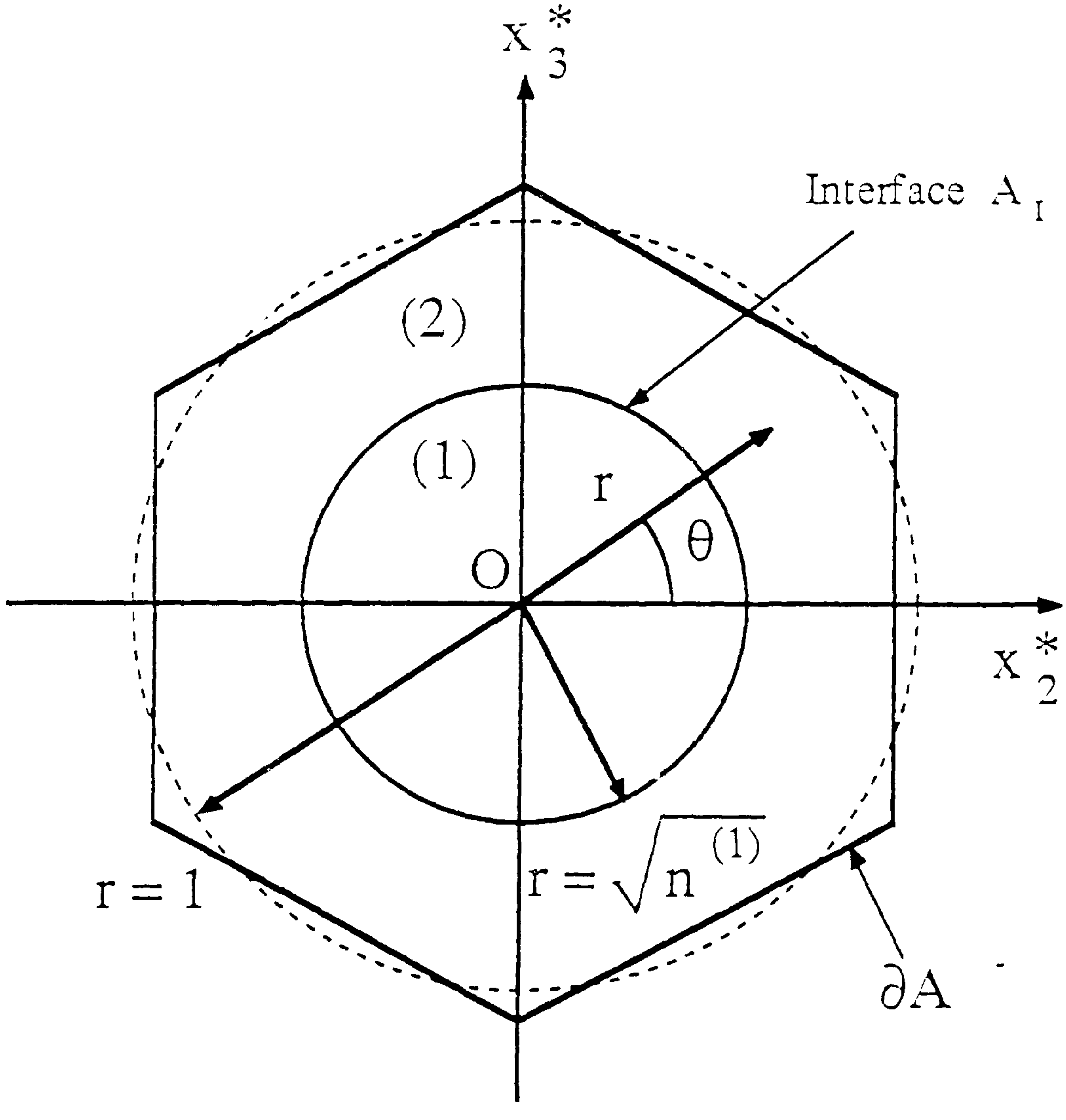

Fig. 2 A unit cell.

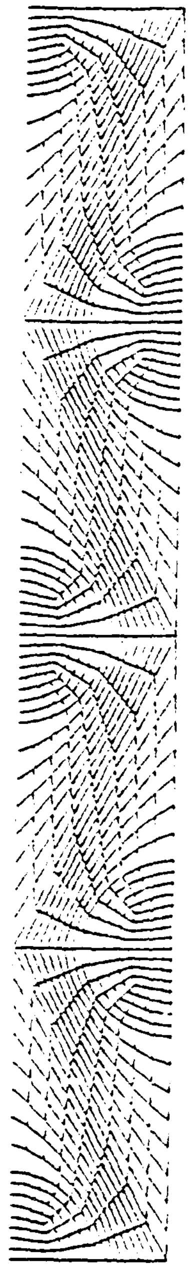

Fig. 3 Finite Element Mesh (NIKE2D)

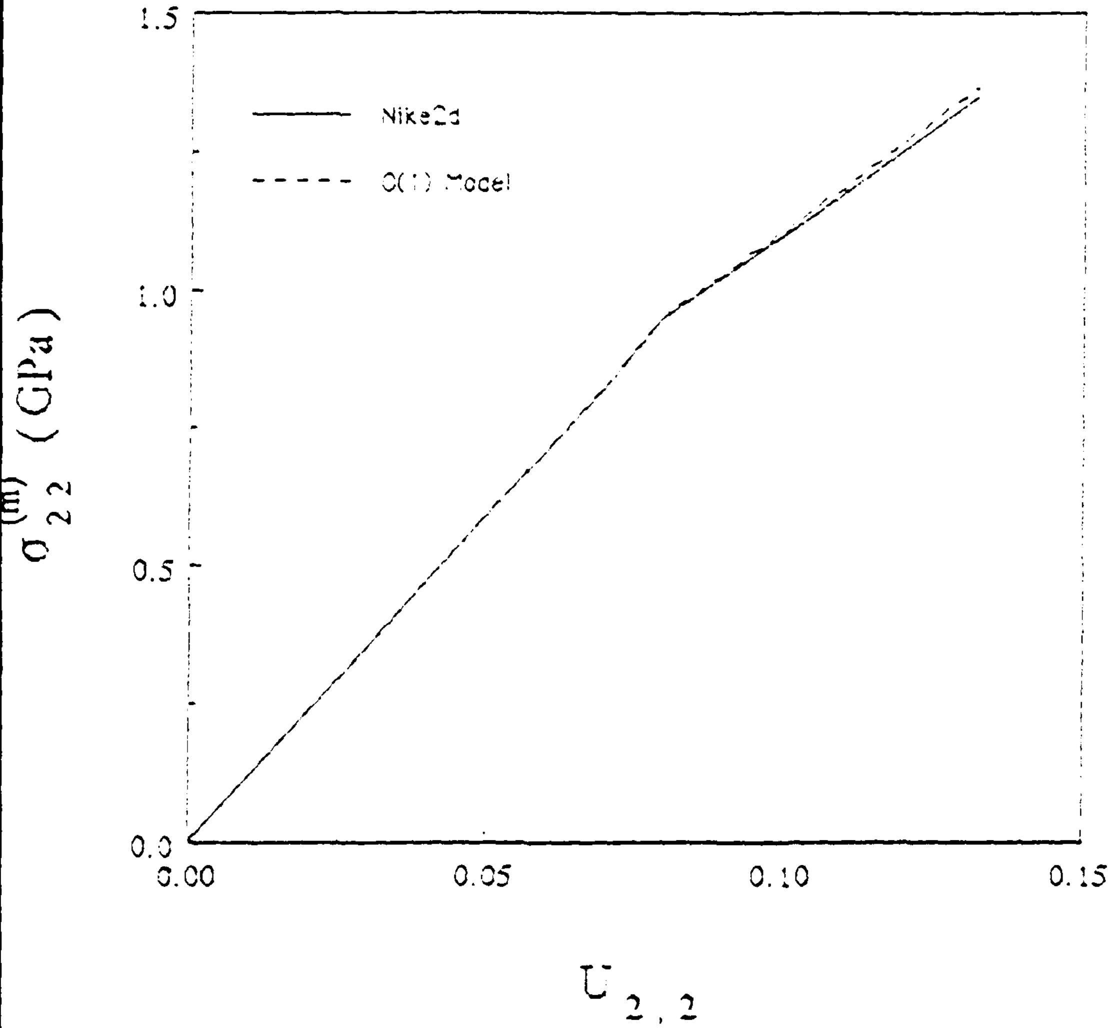

Fig. 4 Effective Stress-Strain Curve

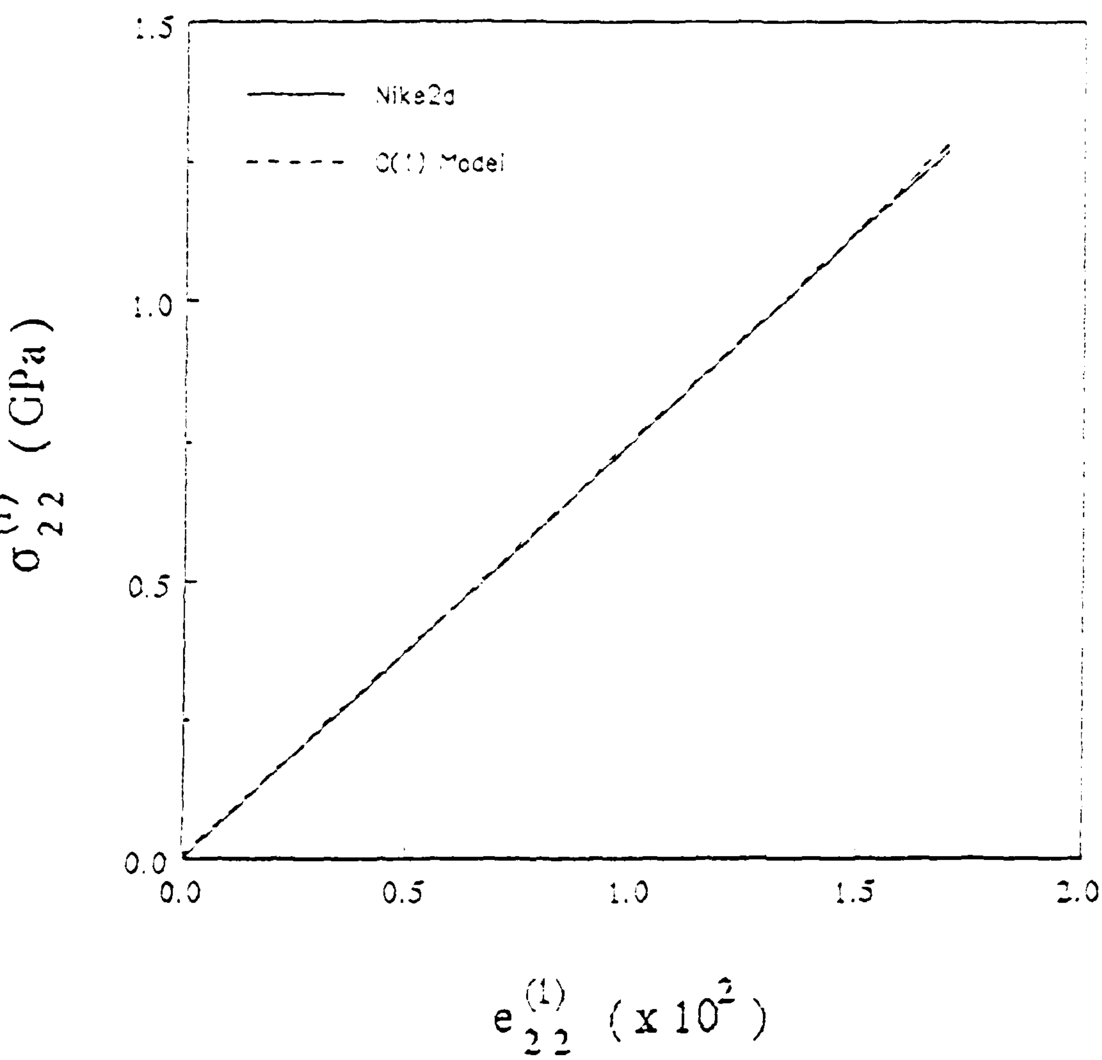

Fig. 5   Fiber Stress-Strain Relation

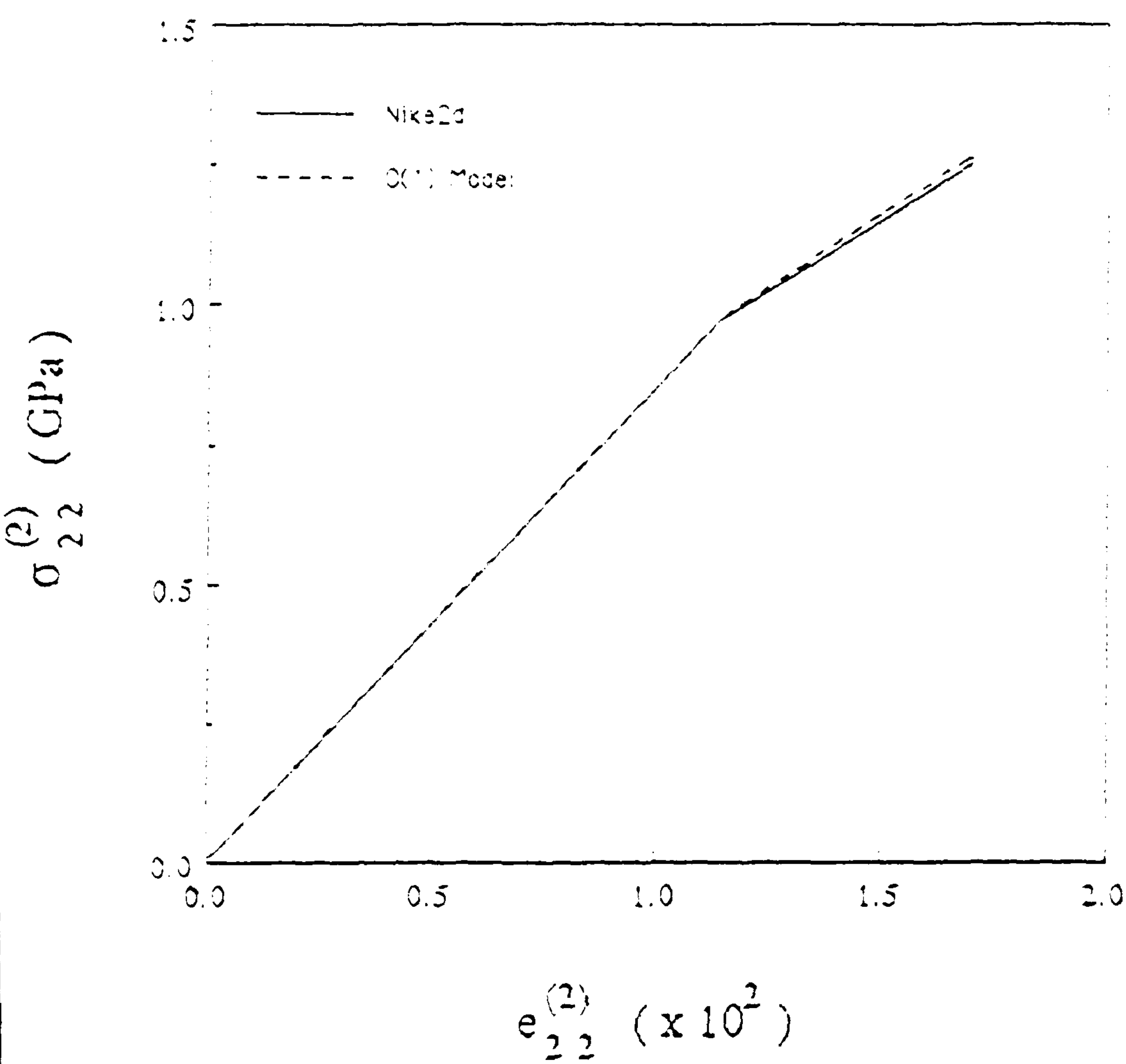

Fig. 6 Matrix stress-strain relation

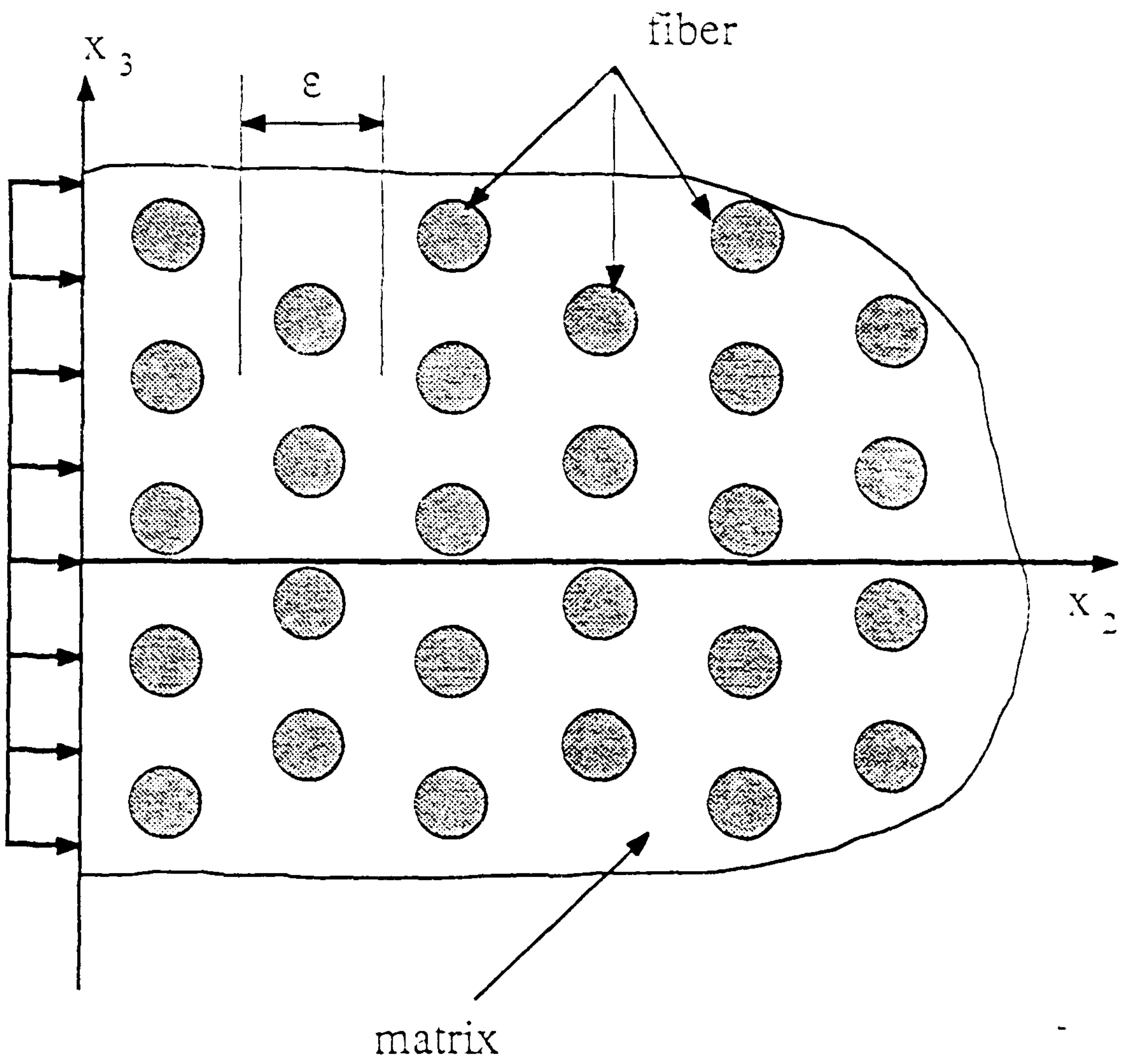

Fig. 7 Loading for wavereflect analysis.

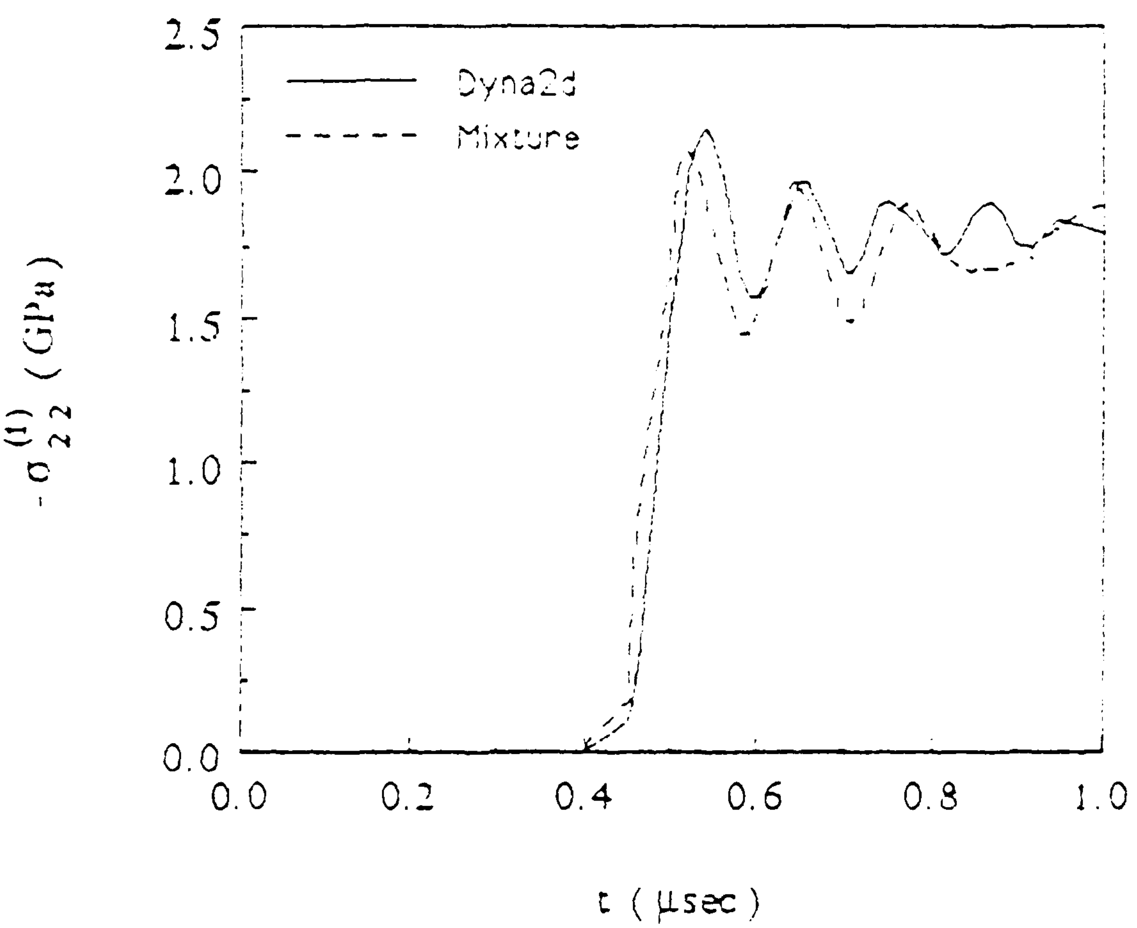

Fig. 8a Time variation of normal stress $\sigma_{22}^{(1)}$ for an elastic analysis

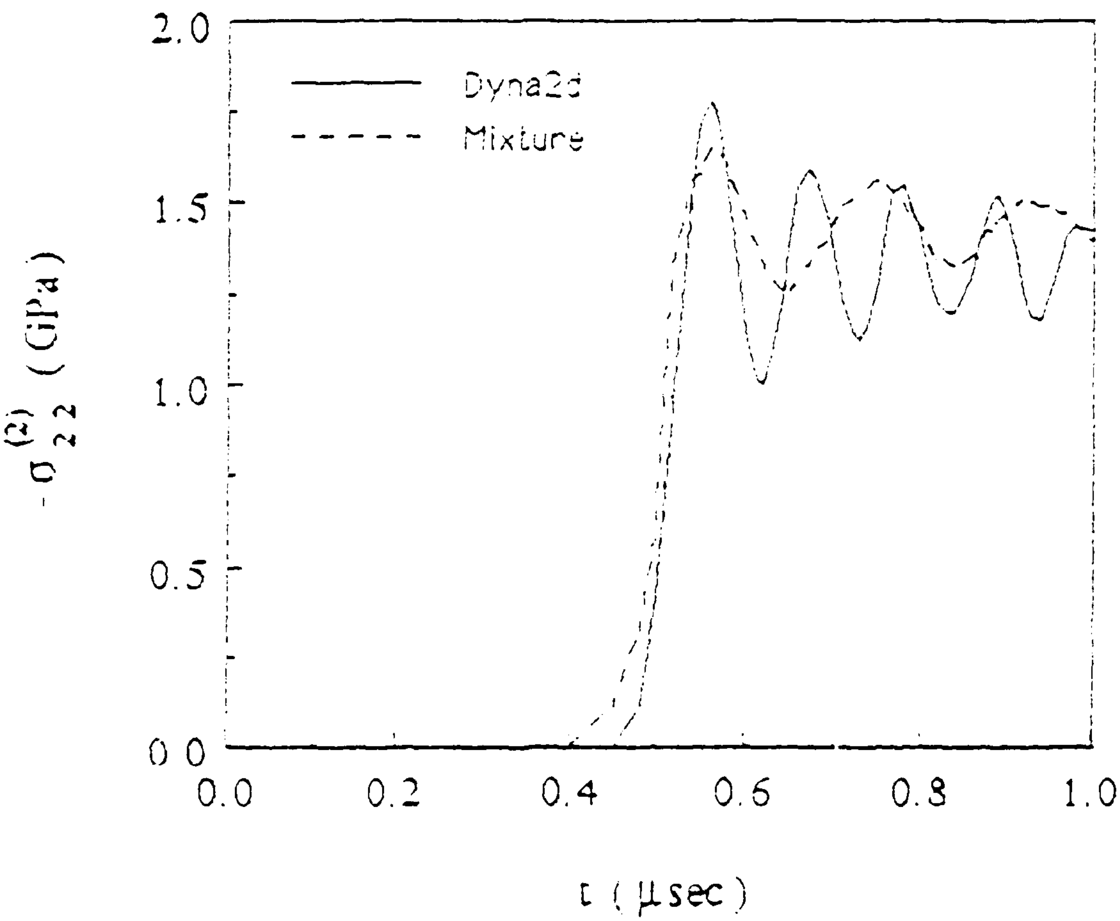

Fig. 8b Time variation of normal stress $\sigma_{22}^{(2)}$ for an elastic analysis

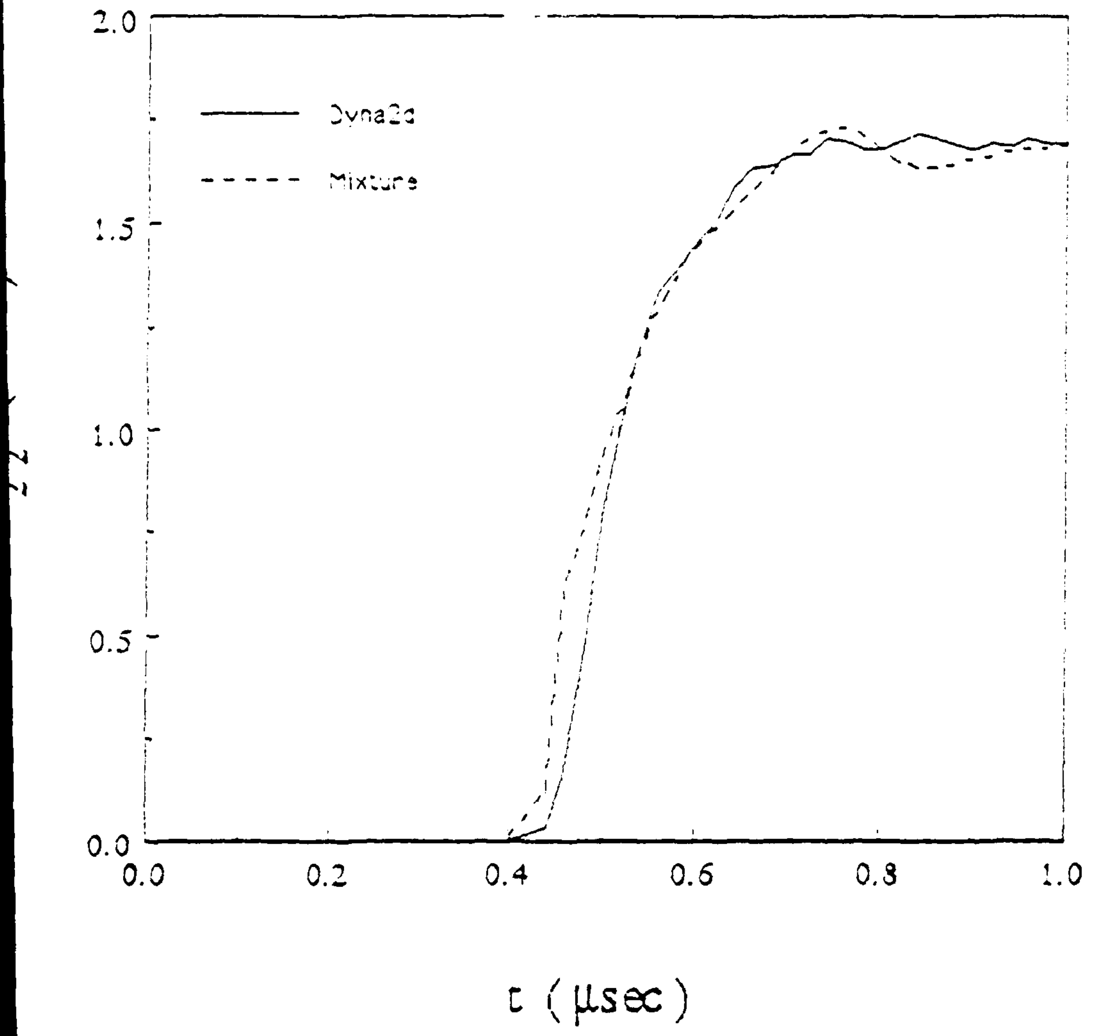

Fig. 9a Time variation of normal stress $\overline{T}_{22}^{(1)}$ for an elastoplastic analysis

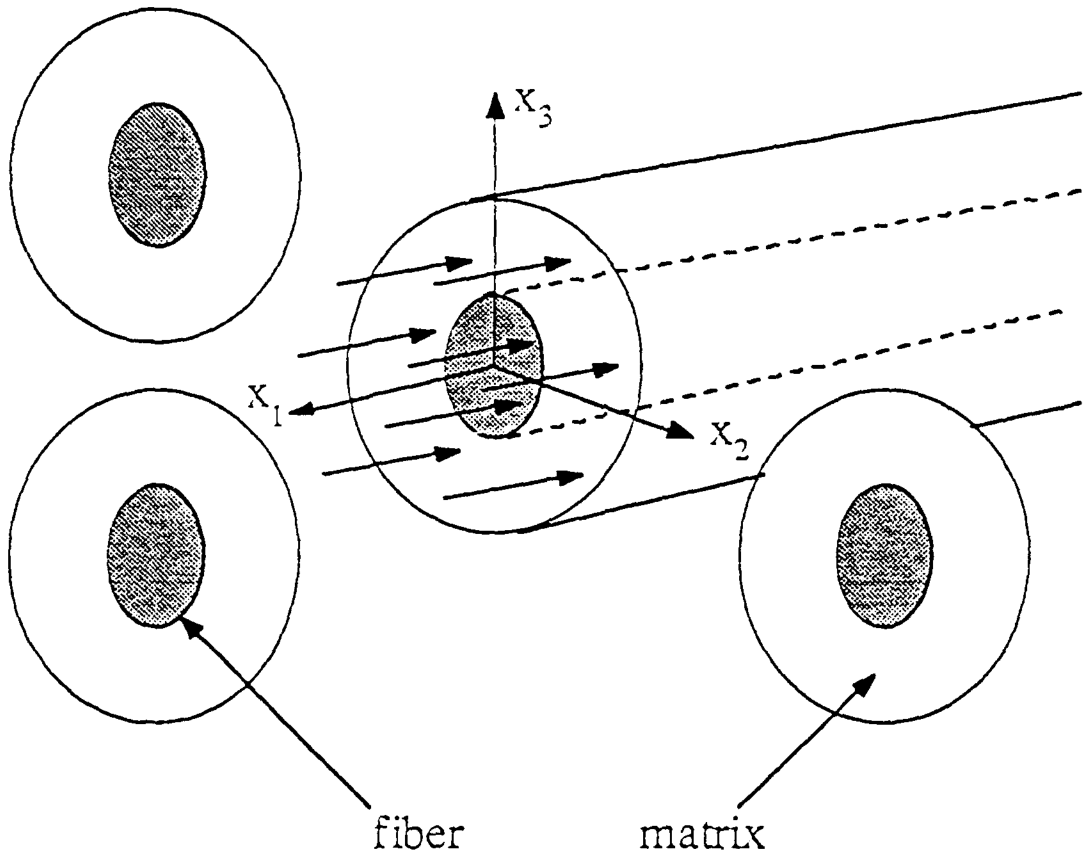

Fig. 10 Loading for waveguide analysis

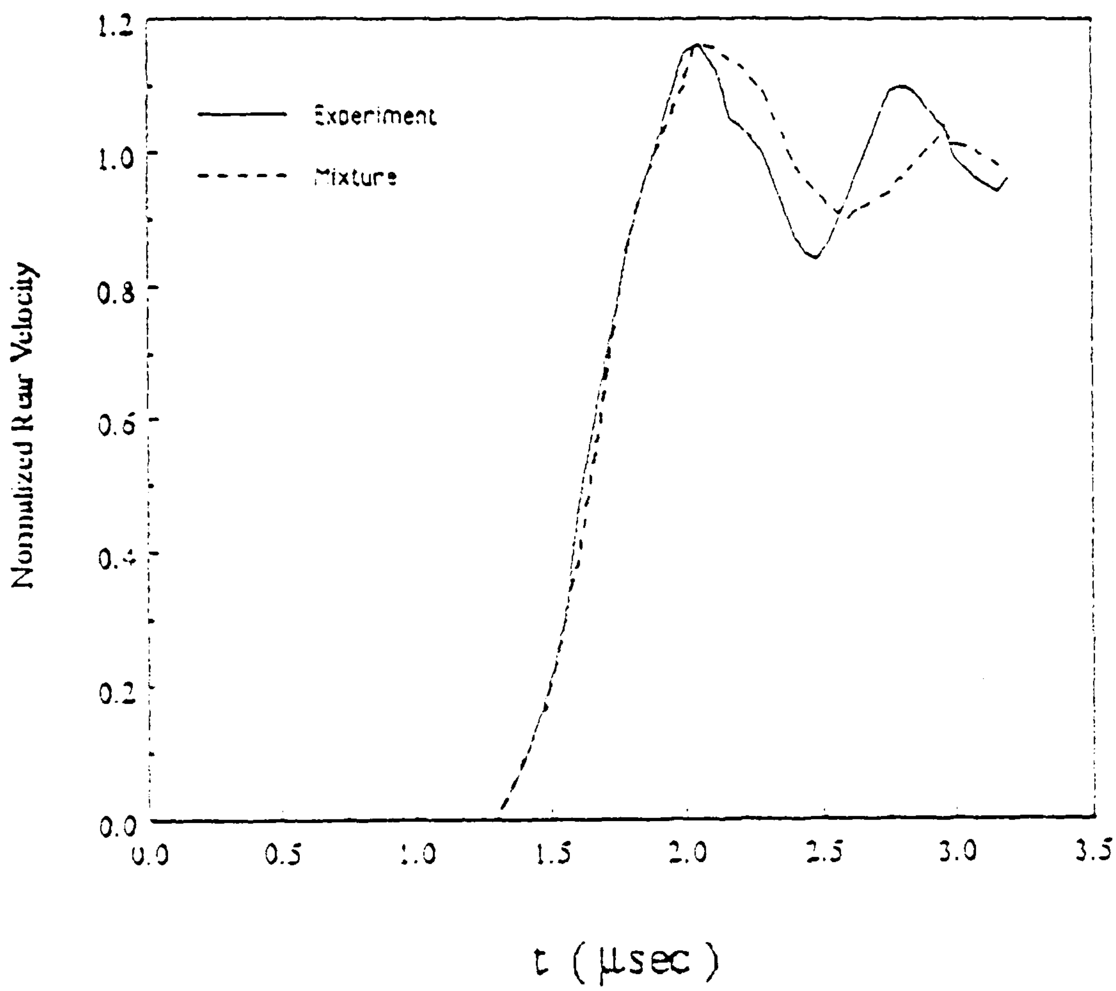

Fig. 11 Time variation of normalized rear velocity

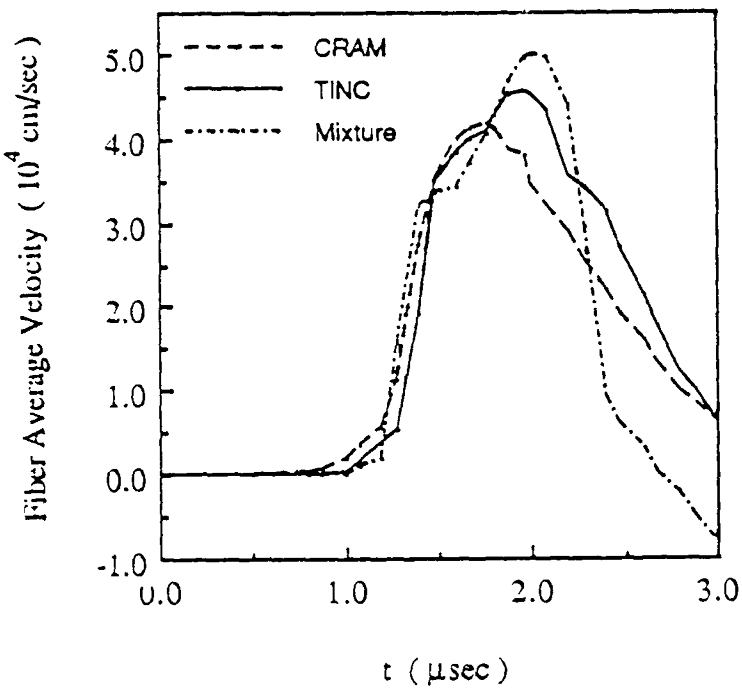

Fig. 12a Time variation of fiber average velocity

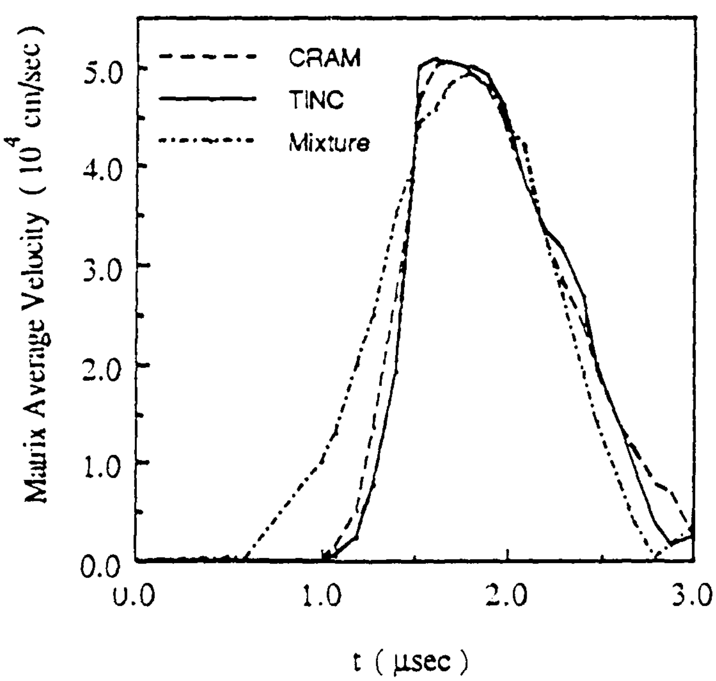

Fig. 12b Time variation of matrix average velocity